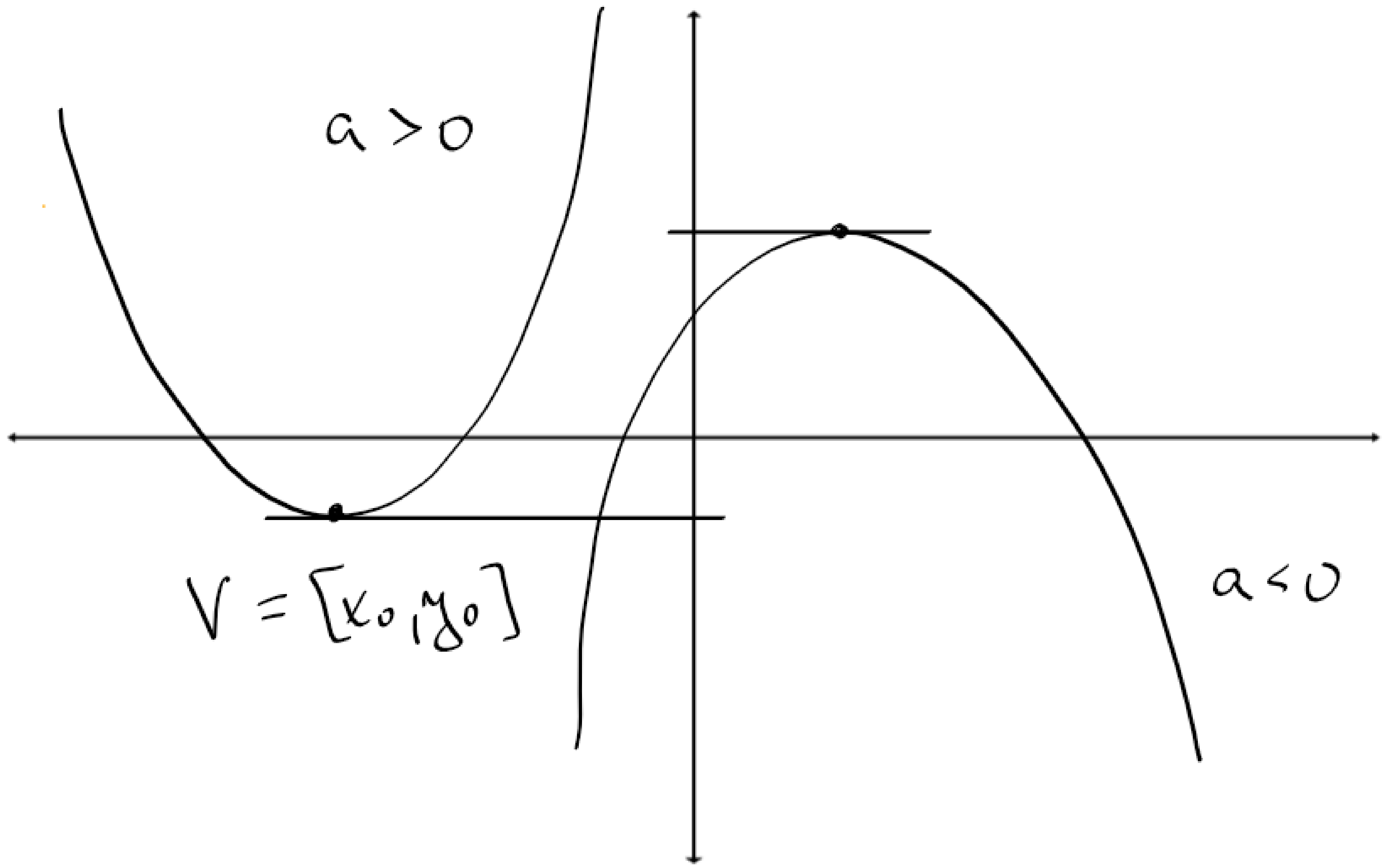
Kvadratické funkce, rovnice a nerovnice

Každá funkce tvaru $y = ax^2 + bx + c$, kde $a, b, c \in \mathbb{R}$, $a \neq 0$, se nazývá kvadratická.

Grafem K.F. je parabola v $\mathbb{R}^2$, definiční obor je $\mathbb{R}$. $f(x) = ax^2+bx+c$,

$$D_f = \mathbb{R}.$$



$$f(x) = ax^2 + bx + c = a\left(x^2 + \frac{b}{a}x\right) + c =$$

$$= a\left(\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a^2}\right) + c =$$

$$\left(x + \frac{b}{2a}\right)^2 = x^2 + 2 \cdot x \cdot \frac{b}{2a} + \frac{b^2}{4a^2} = x^2 + \frac{b}{a}x + \frac{b^2}{4a^2}$$

$$= a\underbrace{\left(x + \frac{b}{2a}\right)^2}_{\geq 0,\ x \in \mathbb{R}} - \frac{b^2}{4a} + c$$

např.: $a > 0$

minimum v bodě $x_0 = -\frac{b}{2a}$

$$f(x_0) = a\left(-\frac{b}{2a} + \frac{b}{2a}\right)^2 - \frac{b^2}{4a} + c =$$

$$= c - \frac{b^2}{4a} = y_0$$

$$V = \left[-\frac{b}{2a},\ c - \frac{b^2}{4a}\right].$$

$$f(x) = ax^2 + bx + c$$

$$f'(x) = 2ax + b = 0 \iff$$

$$x = \frac{-b}{2a} = x_0$$

y_0 dopočítáme jako $y_0 = f(x_0)$.

<u>Příklad:</u> $y = 2x^2 - 4x - 6 = f(x)$

Najděte V paraboly, která je graf f.

$$V = \left[-\frac{b}{2a},\ c - \frac{b^2}{4a}\right] =$$

$$= \left[\frac{4}{4},\ -6 - \frac{16}{8}\right] = [1, -8]$$

$$y = 2x^2 - 4x - 6 = 2\cdot(x^2 - 2x - 3) =$$

$$= 2\cdot\left((x-1)^2 - 1 - 3\right) = 2(x-1)^2 - 8.$$

<u>Kvadratická rovnice</u> je rovnice tvaru

$$ax^2 + bx + c = 0, \quad a, b, c \in \mathbb{R},\ a \neq 0.$$

Počet řešení této rovnice:

$$a\left(\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a^2}\right) + c = 0$$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{c}{a}$$

$$x + \frac{b}{2a} = \pm\sqrt{\frac{b^2}{4a^2} - \frac{c}{a}}$$

$$x = \frac{-b}{2a} \pm \sqrt{\frac{b^2}{4a^2} - \frac{c}{a}}$$

$$= -\frac{b}{2a} \pm \sqrt{\frac{b^2 - 4ac}{4a^2}} =$$

$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

D

diskriminant

Pro $D<0$ ... nemá řešení

$D=0$ ... jediné řešení

$D>0$ ... dvě různá řešení

$D = b^2 - 4ac$ ... diskriminant

Platí $\quad ax^2+bx+c = a(x-x_1)(x-x_2)$,

kde $x_{1,2}$ jsou řešení ax^2+bx+c

Příklad:

$$2x^2-4x-6 = 0$$
$$x^2-2x-3 = 0$$
$$(x-1)^2 = 4$$
$$x-1 = \pm 2$$
$$x = 1 \pm 2 \;\begin{cases} 3 \\ -1 \end{cases}$$

$x_1 = 3, \quad x_2 = -1$

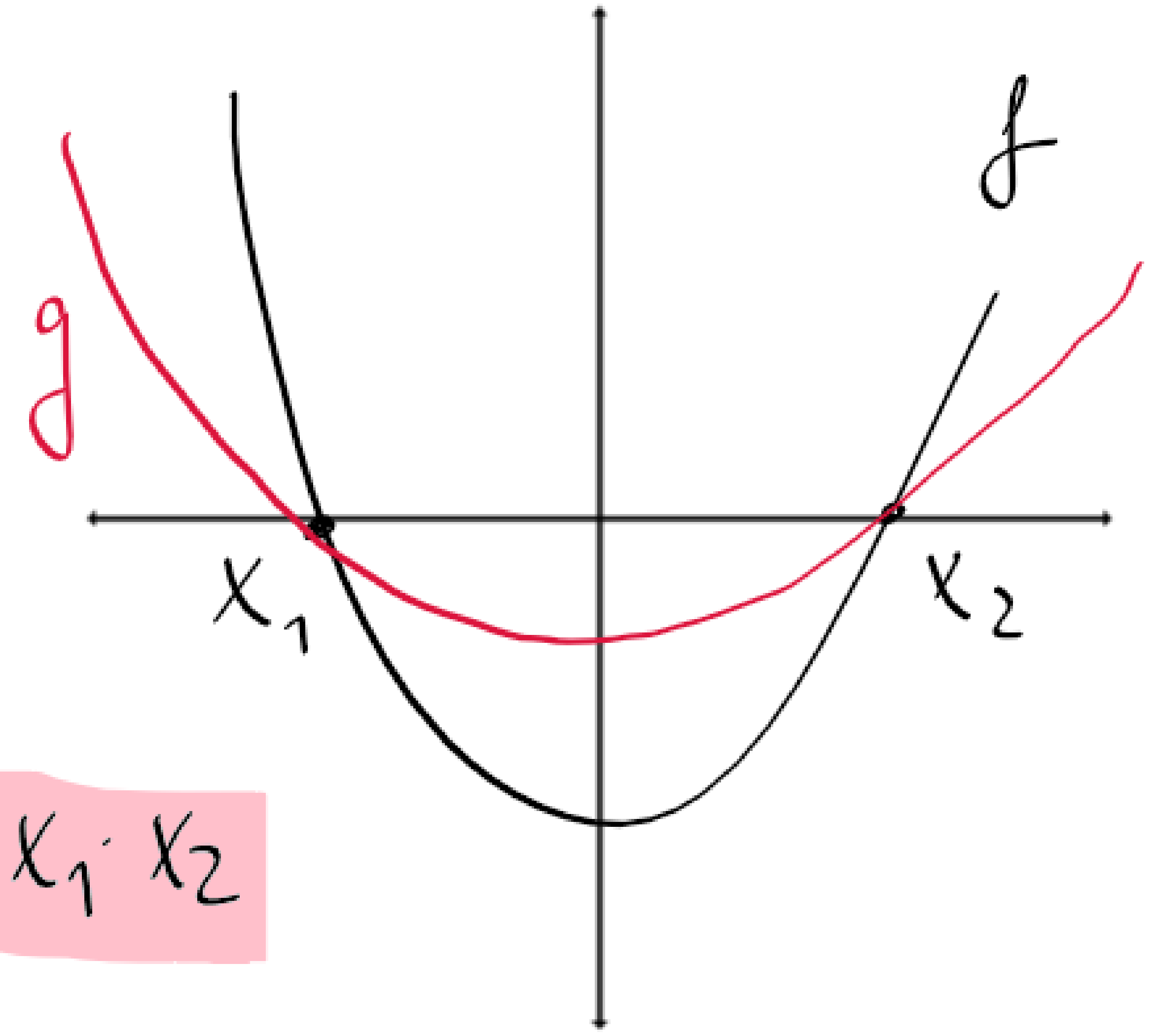
$$2x^2-4x-6 = 2(x-3)(x+1) =$$
$$= 2(x^2-2x-3) = 2x^2-4x-6 \quad \checkmark$$

Z pohledu funkcí:

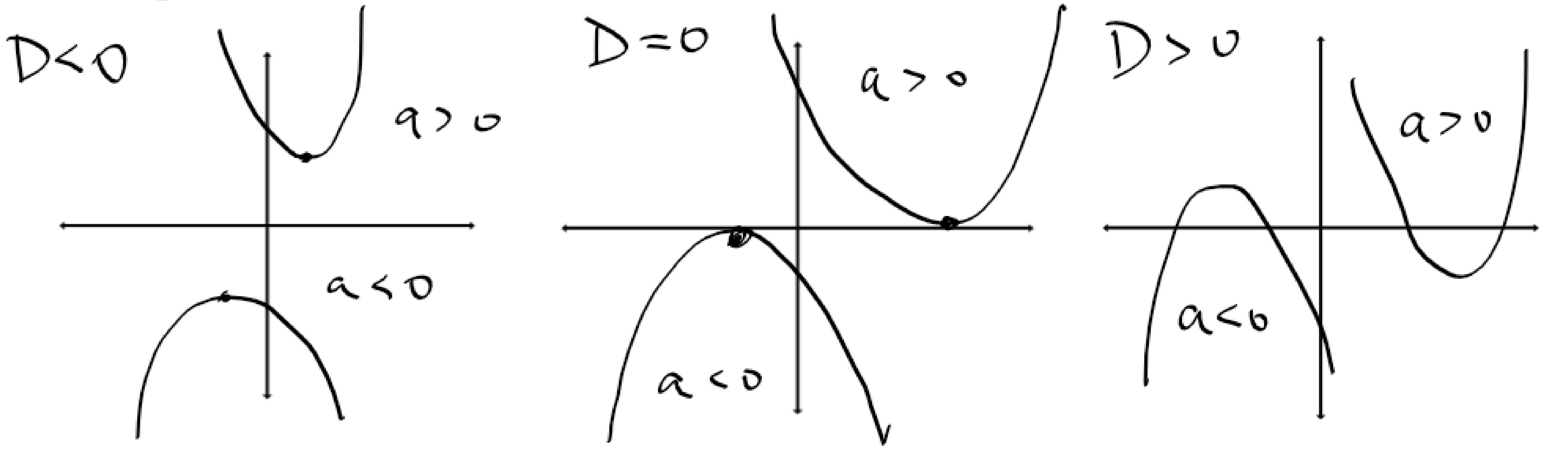
$f(x) = ax^2+bx+c$

$g(x) = (x-x_1)(x-x_2)$

$= x^2-(x_1+x_2)\cdot x + x_1\cdot x_2$



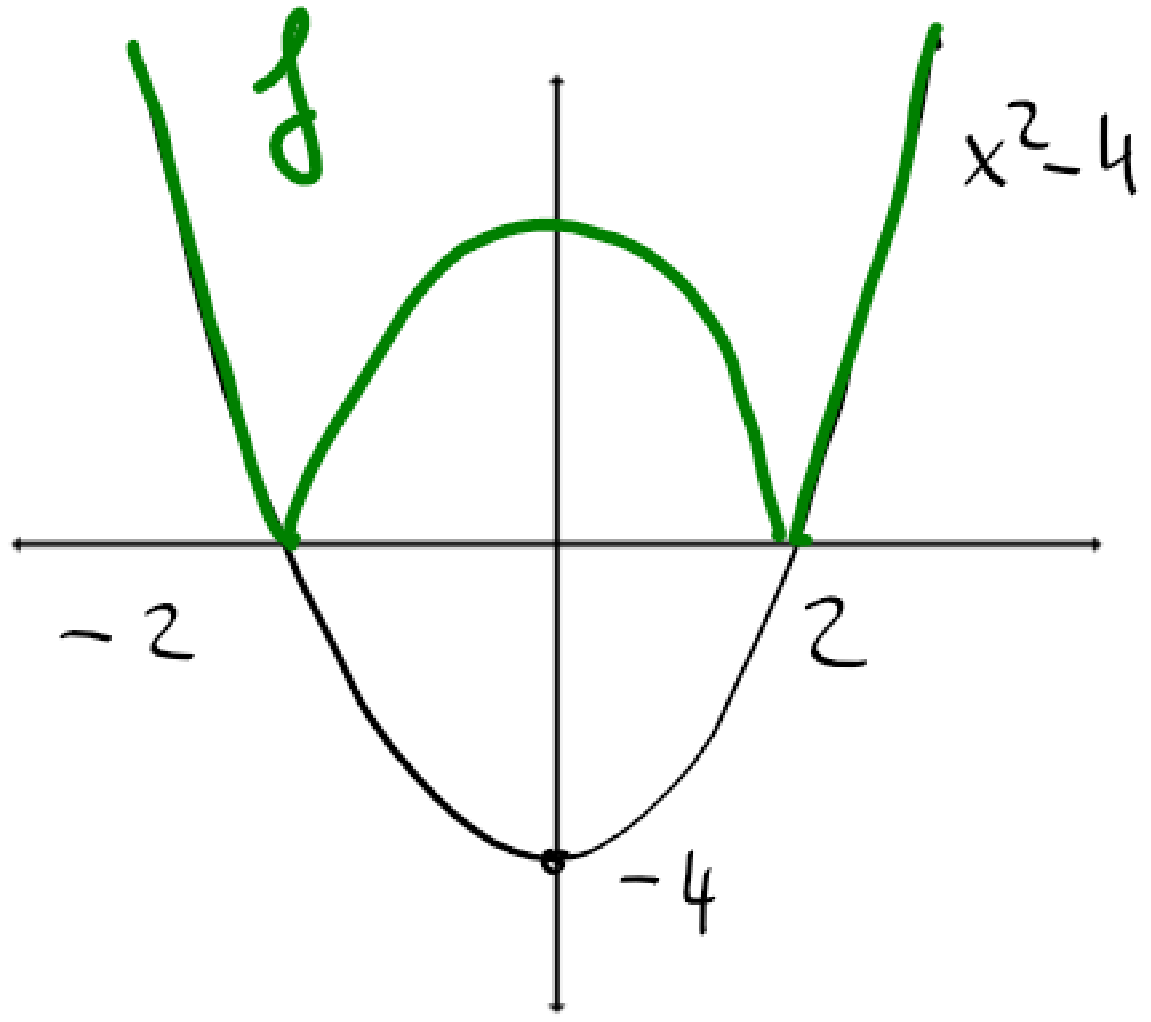
$\frac{f}{g} = a$, neboli $f = a\cdot g$.



Příklad: načrtněte graf $f(x) = \boxed{|x^2-4|}$

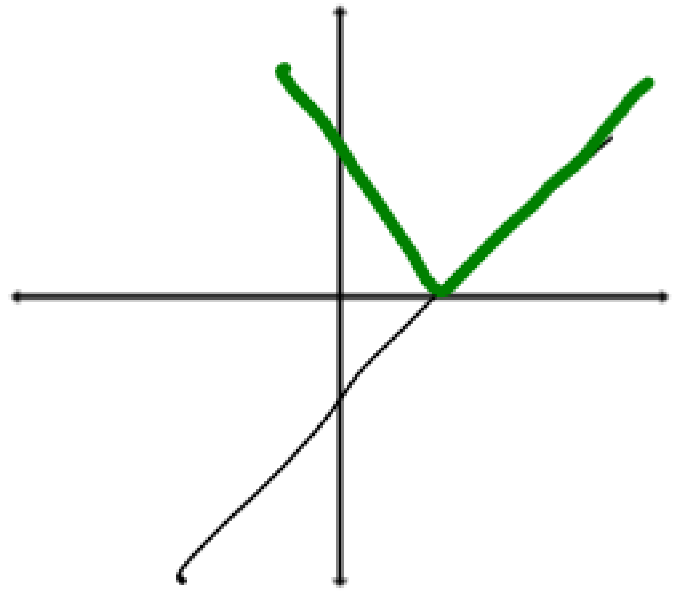
$\boxed{x^2-4} = (x-2)(x+2)$

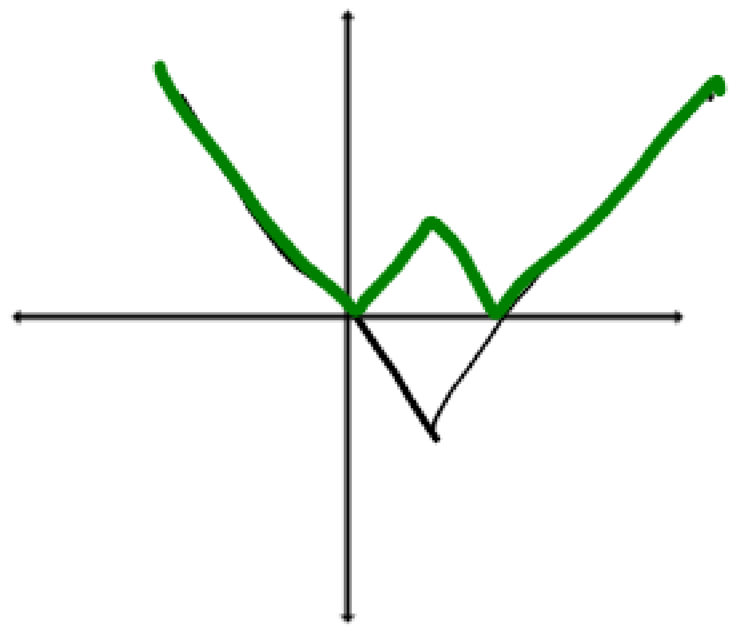
$a^2-b^2 = (a-b)(a+b)$



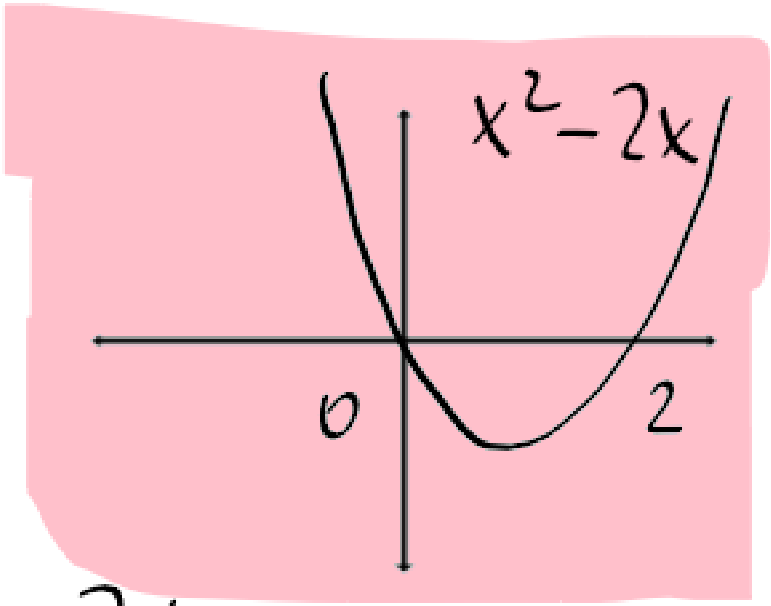
Cv.: Nakreslete graf fce

$$f(x) = ||||x-1|-1|-1|-1|$$





① $2|x^2-2x| + 3x - 3 = 0$



$x^2-2x = x(x-2)$

$x \in (0,2)$: $|x^2-2x| = -x^2+2x$

$2(-x^2+2x) + 3x - 3 = 0$

$-2x^2 + 7x - 3 = 0$

$2x^2 - 7x + 3 = 0$

$$x_{1,2} = \frac{7 \pm \sqrt{49-4\cdot 2\cdot 3}}{4} = \frac{7\pm\sqrt{25}}{4} = \frac{7\pm 5}{4} \quad \cancel{3}, \ \boxed{\frac{1}{2}}$$

$(-\infty,0) \cup \langle 2,\infty)$

$x \in \mathbb{R} \setminus (0,2) = (-\infty,0] \cup [2,\infty)$

$2x^2 - 4x + 3x - 3 = 0$

$\frac{1}{2} - 1 = \frac{3}{2}$

$2x^2 - x - 3 = 0$

$$x_{1,2} = \frac{1 \pm \sqrt{1-4\cdot 2\cdot(-3)}}{4} = \frac{1\pm 5}{4} \quad \cancel{\frac{3}{2}}, \ \boxed{-1}$$

$$2|x^2-2x|+3x-3=0 \iff$$

$$\big(x\in(0,2) \wedge 2(-x^2+2x)+3x-3=0\big) \vee$$

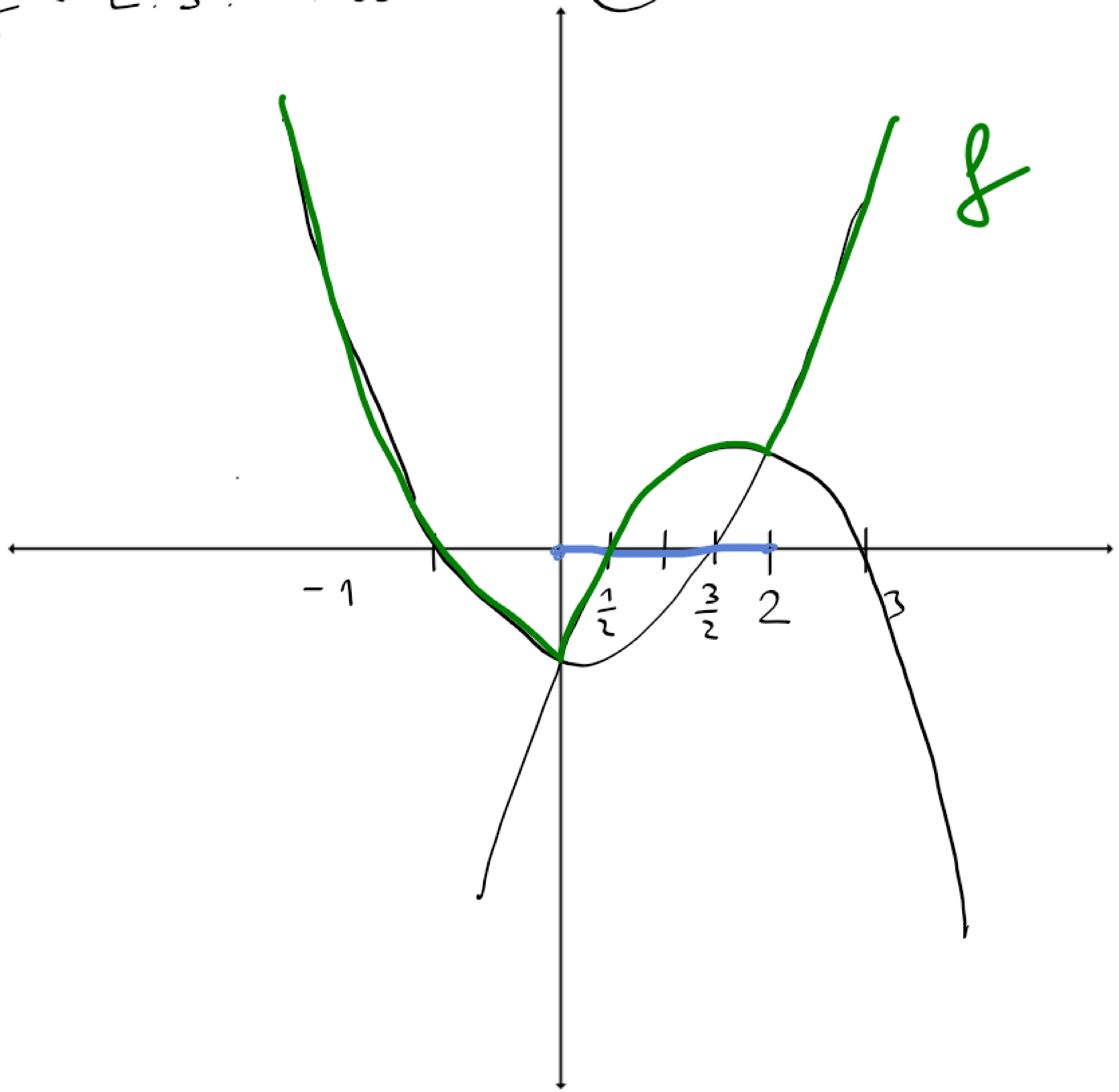
$$\vee \big(x\in\mathbb{R}\setminus(0,2) \wedge 2(x^2-2x)+3x-3=0\big)$$

$$\iff x=\tfrac{1}{2} \vee x=-1$$

$\wedge$ ... a zároveň

$\vee$ .... nebo

f = L.S. rce z (1)



(2) $5x - x^2 \geq 6$ řešte v $\mathbb{R}$.

$$6 - 5x + x^2 \leq 0$$
$$x^2 - 5x + 6 \leq 0$$

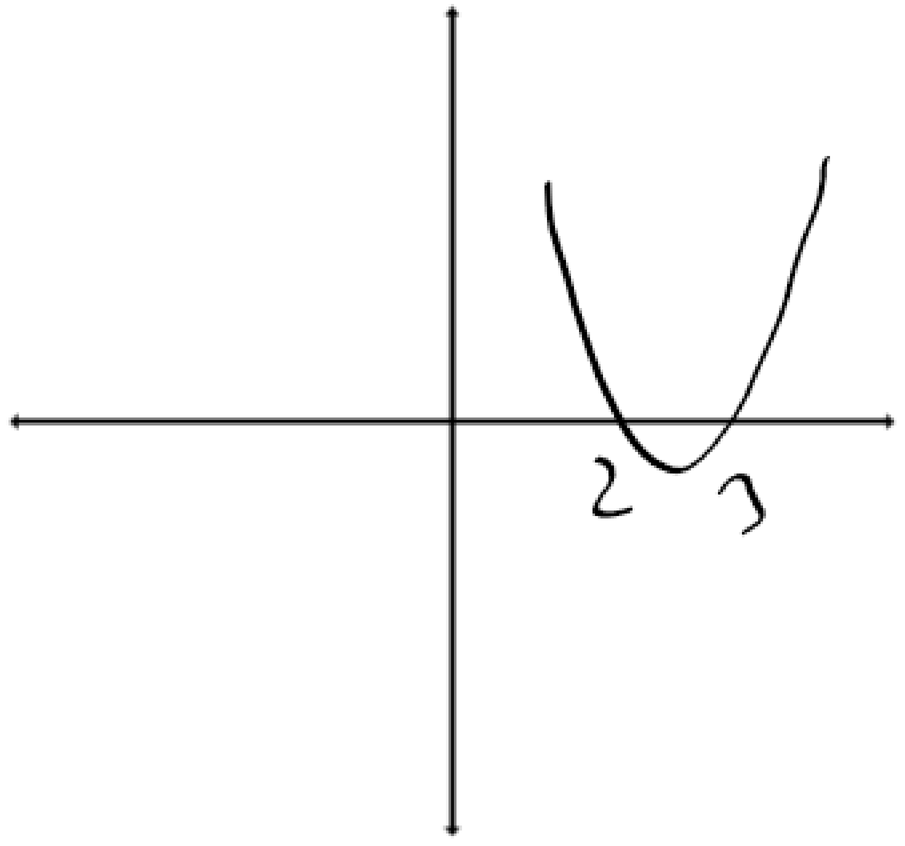
$$x_{1,2} = \frac{5 \pm \sqrt{25-24}}{2} = \frac{5 \pm 1}{2} = \begin{cases} 3 \\ 2 \end{cases}$$

$$x^2 - 5x + 6 = (x-3)(x-2)$$

$$(x-3)(x-2) \leq 0$$

$$\iff$$

$$x \in [2,3]$$

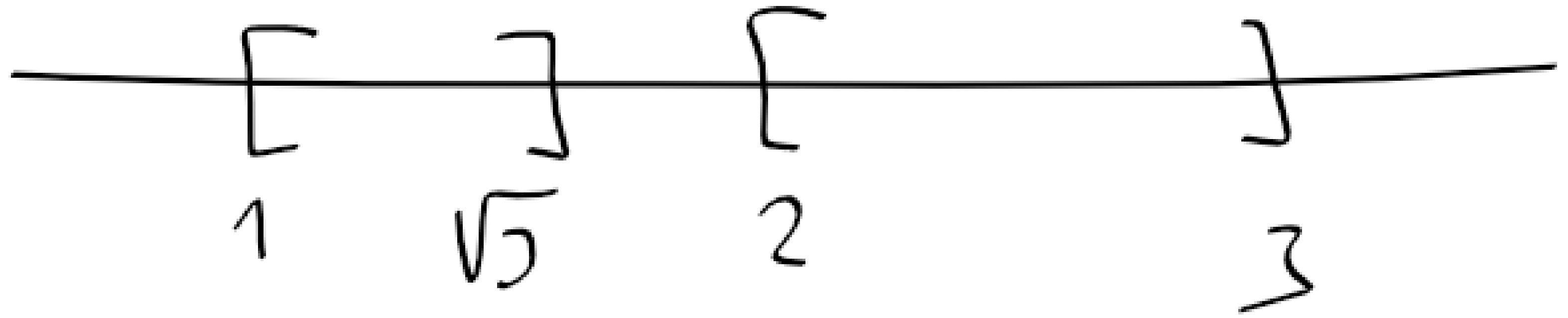


- $[2,3] \subseteq (0,\infty)$
- $[2,3] \not\subseteq (-\infty,0)$ $\sqrt{2} < 1{,}5 \quad /^2$
- $2\sqrt{2} < 3$ $2 < 2{,}25$

$\implies \exists x \in [2,3] : x > 2\sqrt{2}$ (třeba $x=3$)

$M = [2,3] \overset{?}{\subseteq} \langle \sqrt[3]{2}, \frac{22}{7} \rangle$ ✓

$M \cap \langle 1, \sqrt{3} \rangle \neq \emptyset$ ✗



(3) $x^2 - 5 - |x+1| > 0$

$\underline{x < -1}$: $x^2 - 5 - (-x-1) > 0$ ....

$\underline{x \geq -1}$: $x^2 - 5 - (x+1) > 0$ --

(4.) $\dfrac{x-x^2}{x^2+x-2} \leq 0 \iff x \in M$

Čitatel: $x - x^2 = x(1-x)$ $\quad 0, 1$

Jmenovatel: $x^2 + x - 2 = 0$

$$x_{1,2} = \frac{-1 \pm \sqrt{1-4\cdot(-2)}}{2} = \frac{-1 \pm 3}{2} \begin{cases} 1 \\ -2 \end{cases} \quad 1, -2$$

$$x^2 + x - 2 = (x-1)(x+2)$$

		-2		0		1	
čit	$-$		$-$		$+$		$-$
jm.	$+$		$-$		$-$		$+$

$$M = (-\infty, -2) \cup [0, 1) \cup (1, \infty)$$

(5.) $\dfrac{x^2+4x-77}{x-7} \leq 2$

Postup A: $\dfrac{x^2+4-77-2(x-7)}{x-7} \leq 0$

viz předchozí př.

Postup B: $\dfrac{x^2+4x-77}{(x-7)} \leq 2 \quad / \cdot (x-7)$

$x > 7$: $\quad x^2 + 4x - 77 \leq 2(x-7)$

řešíme dál

$x < 7$: $\quad x^2 + 4x - 77 \geq 2(x-7)$

řešíme dál

$$\frac{x^2+4x-77}{x^2+1} \leq 2 \iff$$

$$\iff x^2+4x-77 \leq 2(x^2+1)$$

(pro $x \in \mathbb{R}$), protože $\forall x \in \mathbb{R}: x^2+1>0$

$$\frac{x^2+4x-77}{x^2+4x+5} \leq 2 \iff$$

$$\boxed{\begin{array}{l} D = 16-4\cdot 5 = 16-20 < 0 \\ \text{Tedy } x^2+4x+5>0, \; x \in \mathbb{R} \end{array}}$$

$$\iff x^2+4x-77 \leq 2(x^2+4x+5)$$